Exponent and Logarithm Practice Problems for Precalculus and Calculus

1. Expand $(x+y)^5$.
2. Simplify the following expression:
$$\left(\frac{b^3\sqrt{5b+2}}{a-b}\right)^2.$$
3. Evaluate the following powers: $13^0 =$, $(-8)^{2/3} =$, $5^{-2} =$, $81^{-1/4} =$
4. Simplify $\left(\frac{243y^{10}}{32z^{15}}\right)^{-2/5}$.
5. Simplify $\left(\frac{42(3a+1)^6}{7(3a+1)^{-1}}\right)^2$.
6. Evaluate the following logarithms: $\log_5 125 =$,$\log_4 \frac{1}{2} =$, $\log 1000000 =$, $\log_b 1 =$, $\ln(e^x) =$
7. Simplify: $\frac{1}{2}\log(x) + \log(y) - 3\log(z)$.
8. Evaluate the following: $\log(\sqrt{10}\sqrt[3]{10}\sqrt[5]{10}) =$, $1000^{\log 5} =$, $0.01^{\log 2} =$
9. Write as sums/differences of simpler logarithms without quotients or powers
$$\ln\left(\frac{e^3x^4}{e}\right).$$
10. Solve for x: $3^{x+5} = 27^{-2x+1}$.
11. Solve for x: $\log(1-x) - \log(1+x) = 2$.
12. Find the solution of: $\log_4(x-5) = 3$.
13. What is the domain and what is the range of the exponential function $y = ab^x$ where a and b are both positive constants and $b \neq 1$?
14. What is the domain and what is the range of $f(x) = \log(x)$?
15. Evaluate the following expressions.
    (a) $\ln(e^4) =$
    (b) $\sqrt{\log(10000)} - \log\sqrt{100} =$
    (c) $e^{\ln(3)} =$
    (d) $\log(\log(10)) =$
16. Suppose $x = \log(A)$ and $y = \log(B)$, write the following expressions in terms of x and y.
    (a) $\log(AB) =$
    (b) $\log(A)\log(B) =$
    (c) $\log\left(\frac{A}{B^2}\right) =$

Solutions

1. We can either do this one by "brute force" or we can use the *binomial theorem* where the coefficients of the expansion come from Pascal's triangle. Either way, the solution is

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

2. We first use the fact that when raising a fraction to a power we raise the numerator and denominator to that same power and the fact that when we raise a product to a power we raise the individual factors to that same power to say

$$\left(\frac{b^3\sqrt{5b+2}}{a-b}\right)^2 = \frac{(b^3)^2(\sqrt{5b+2})^2}{(a-b)^2}.$$

Next we actually do the squaring to get

$$\frac{(b^3)^2(\sqrt{5b+2})^2}{(a-b)^2} = \frac{b^6(5b+2)}{a^2-2ab+b^2} = \frac{5b^7+2b^6}{a^2-2ab+b^2}$$

and we are done.

3. By definition, $13^0 = 1$, $(-8)^{2/3} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$ (or this could be done as $(-8)^{2/3} = \left(\sqrt[3]{-8}\right)^2 = (-2)^2 = 4$), $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$, $81^{-1/4} = \frac{1}{\sqrt[4]{81}} = \frac{1}{3}$.

4. First we can use the fact that if you raise a fraction to a negative power, that is the same as raising the reciprocal of that fraction to the opposite of that power. In other words,

$$\left(\frac{243y^{10}}{32z^{15}}\right)^{-2/5} = \left(\frac{32z^{15}}{243y^{10}}\right)^{2/5}.$$

Next we use the fact that when raising a fraction to a power we raise the numerator and denominator to that same power and the fact that when we raise a product to a power we raise the individual factors to that same power to say

$$\left(\frac{32z^{15}}{243y^{10}}\right)^{2/5} = \frac{\left(32z^{15}\right)^{2/5}}{\left(243y^{10}\right)^{2/5}} = \frac{32^{2/5}(z^{15})^{2/5}}{243^{2/5}(y^{10})^{2/5}}.$$

Now, by definition, raising a number to the 2/5 power is the same as squaring its fifth root (or taking the fifth root of its square). Thus, we get

$$\frac{32^{2/5}(z^{15})^{2/5}}{243^{2/5}(y^{10})^{2/5}} = \frac{4z^6}{9y^4}$$

and we are done.

5. To simplify $\left(\frac{42(3a+1)^6}{7(3a+1)^{-1}}\right)^2$, we first use the facts that $\frac{42}{7} = 6$ and $\frac{(3a+1)^6}{(3a+1)^{-1}} = (3a+1)^{6-(-1)} = (3a+1)^7$ to write

$$\left(\frac{42(3a+1)^6}{7(3a+1)^{-1}}\right)^2 = (6(3a+1)^7)^2 = 6^2(3a+1)^{14} = 36(3a+1)^{14}.$$

As it stands, this is pretty simple. However, we could also expand it by multiplying out $(3a+1)^{14}$ using the binomial theorem. We'll spare you the pain of trying that.

6. We use the definition of the quantity $\log_b a$ as being the number which you must raise b to in order to get a (when $a > 0$). In other words, $b^{\log_b a} = a$ by definition. So, $\log_5 125 = 3$ since $5^3 = 125$,$\log_4 \frac{1}{2} = -\frac{1}{2}$ since $4^{-1/2} = \frac{1}{2}$, $\log 1000000 = 6$ since $10^6 = 1000000$, $\log_b 1 = 0$ since $b^0 = 1$, $\ln(e^x) = x$ since $e^x = e^x$ ($\ln(a)$ means log base-e of a, where $e \approx 2.718$).

7. To simplify the expression $\frac{1}{2}\log(x) + \log(y) - 3\log(z)$, we must use some fundamental properties of logarithms (which work no matter what the base is...in this case we are using base 10). The first such property we make use of is the fact that $r\log(x) = \log(x^r)$ for all $x > 0$ to say that $\frac{1}{2}\log(x) = \log(x^{1/2}) = \log(\sqrt{x})$ and $3\log(z) = \log(z^3)$ (where we are implicitly assuming that $x > 0$ and $z > 0$). Therefore,

$$\frac{1}{2}\log(x) + \log(y) - 3\log(z) = \log(\sqrt{x}) + \log(y) - \log(z^3).$$

Next, we use the fact that the log of a product is the sum of the logs and the log of a quotient is the difference of the logs. To be more specific, we use the facts that $\log(ab) = \log(a) + \log(b)$ and $\log(\frac{a}{b}) = \log(a) - \log(b)$ (where $a > 0$ and $b > 0$). Because of these facts, we can write:

$$\log(\sqrt{x}) + \log(y) - \log(z^3) = \log(y\sqrt{x}) - \log(z^3) = \log\left(\frac{y\sqrt{x}}{z^3}\right)$$

and we are done.

8. Using properties of logs and exponents, $\log(\sqrt{10}\sqrt[3]{10}\sqrt[5]{10}) = \log(10^{1/2})+\log(10^{1/3})+\log(10^{1/5}) = \frac{1}{2}+\frac{1}{3}+\frac{1}{5} = \frac{15}{30}+\frac{10}{30}+\frac{6}{30} = \frac{31}{30}$, $1000^{\log 5} = \left(10^3\right)^{\log 5} = 10^{3\log 5} = 10^{\log 5^3} = 5^3 = 125$, $0.01^{\log 2} = \left(10^{-2}\right)^{\log 2} = 10^{-2\log 2} = 10^{\log(2^{-2})} = 2^{-2} = \frac{1}{4}$.

9. Here we use the same rules as in problem 7, but in the "other direction" (also, we have natural logs here instead of common logs). We can write:

$$\ln\left(\frac{e^3x^4}{e}\right) = \ln(e^3x^4) - \ln(e) = \ln(e^3) + \ln(x^4) - \ln(e) = 3 + 4\ln(x) - 1 = 2 + 4\ln(x)$$

and we are done.

10. We could solve the equation $3^{x+5} = 27^{-2x+1}$ using logarithms, but this is unnecessary because $27 = 3^3$. Because of this fact, our equation is equivalent to $3^{x+5} = (3^3)^{-2x+1} = 3^{-6x+3}$. Now the fact that $f(x) = 3^x$ is a one-to-one function implies that $x + 5 = -6x + 3$. This is now a linear equation in x which can be solved by isolating x to get $7x = -2$ and so $x = -\frac{2}{7}$.

Technically we should check this by plugging it into the original equation. If we do so, on the left-hand side we get $3^{5-\frac{2}{7}} = 3^{33/7}$ and on the right-hand side we get $27^{\frac{4}{7}+1} = 27^{11/7} = 3^{33/7}$, so it works.

11. First we write $\log(1-x)-\log(1+x) = 2$ as $\log\left(\frac{1-x}{1+x}\right) = 2$. This means that $\frac{1-x}{1+x} = 10^2 = 100$ so that $1-x = 100(1+x) = 100+100x$. Thus, $101x = -99$ and $x = -\frac{99}{101}$. Checking this in the original equation gives

$$\log\left(1 - \left(-\frac{99}{101}\right)\right) - \log\left(1 + \left(-\frac{99}{101}\right)\right) = \log\left(\frac{200}{101}\right) - \log\left(\frac{2}{101}\right) = \log\left(\frac{200}{101}\cdot\frac{101}{2}\right) = \log\left(\frac{200}{2}\right) = \log(100) = 2.$$

12. The equation $\log_4(x-5) = 3$ can be rewritten as $x - 5 = 4^3 = 64$. This means that $x = 69$.

13. The domain (the set of all allowed inputs) of the function $y = ab^x$ where a and b are positive constants and $b \neq 1$ is the set of all real numbers, in symbols, the set $\mathbb{R} = (-\infty, \infty) = \{x | x \text{ is a real number}\}$.

The range (the set of all possible outputs as x ranges over the domain) of this function is the set of all positive real numbers, in symbols, the set $(0, \infty) = \{y | y \text{ is a real number} > 0\}$.

14. The domain of the function $f(x) = \log(x)$ is the set of all positive real numbers and the range is the set of all real numbers.

15. Using properties of logarithms, we can write

(a) $\ln(e^4) = 4$

(b) $\sqrt{\log(10000)} - \log\sqrt{100} = \sqrt{4} - \log 10 = 2 - 1 = 1$

(c) $e^{\ln(3)} = 3$

(d) $\log(\log(10)) = \log(1) = 0$

16. Using properties of logarithms, we can write

(a) $\log(AB) = \log(A) + \log(B) = x + y$

(b) $\log(A)\log(B) = xy$

(c) $\log\left(\frac{A}{B^2}\right) = \log(A) - 2\log(B) = x - 2y$